Controlling Entanglement Generation in External Quantum Fields

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Abstract

Two, non-interacting two-level atoms immersed in a common bath can become mutually entangled when evolving with a Markovian, completely positive dynamics. For an environment made of external quantum fields, this phenomenon can be studied in detail: one finds that entanglement production can be controlled by varying the bath temperature and the distance between the atoms. Remarkably, in certain circumstances, the quantum correlations can persist in the asymptotic long-time regime.

1 Introduction

Independent atoms immersed in external quantum fields and weakly coupled to them can be viewed as open systems, *i.e.* as subsystems in interaction with an environment [1]-[4]. The atoms can be usually treated in a non-relativistic approximation, as independent n-level systems, with negligible size, while the environment is described by a set of quantum fields (*e.g.* the electromagnetic field) in a given quantum state, typically either a temperature state or simply the vacuum state. The interaction of the fields with the atoms is taken to be of dipole type, a well justified approximation within the weak coupling assumption [5].

Even in this simplified setting, that ignores all intricancies related to the internal atom structure and the full coupling with the electromagnetic field, the model is of great relevance both theoretically and phenomenologically [5]-[7]: indeed, with suitable adaptations, it is able to capture the main features of the dynamics of very different physical systems, like ions in traps, atoms in optical cavities and fibers, impurities in phonon fields.

Despite this ample range of possible applications and the attention devoted to them in the recent literature, no particular care has often been taken in the derivation of an acceptable

subdynamics for the atoms. As a result, time evolutions that are not even positive, have been adopted in order to describe their physical properties.

On the other hand, a mathematically sound and physically consistent time evolution for the atom subsystem can be obtained using the *weak coupling limit* procedure [8]: the resulting subdynamics is described by a one parameter ($\equiv$ time) family of completely positive maps that form a quantum dynamical semigroup.

In the following we shall outline such a derivation, and apply the resulting dynamics to the study of the evolution of a system composed by two, independent atoms. For simplicity we shall restrict the attention to two-level atoms in interaction with a collection of independent, free, massless scalar fields in $3 + 1$ space-time dimensions, assumed to be in a state at temperature $T = 1/\beta$.

The interaction with an environment usually leads to decoherence and noise, typical mixing enhancing phenomena. Therefore, one generally expects that when a bipartite system is immersed in an environment, quantum correlations that might have been created before by a direct interaction between the two subsystems actually disappear.

However, an external environment can also provide an indirect interaction between otherwise totally decoupled subsystems and therefore a mean to correlate them [9]-[14],[6]. This phenomenon has first been established in exactly solvable models [9]: there, correlations between the two subsystems take place during a short time transient phase, where the reduced dynamics of the subsystems contains memory effects.

Remarkably, entanglement generation may also occur in the Markovian regime, through a purely noisy mechanism [15, 16]. It is precisely this situation that is relevant in the analysis of the dynamics of two independent atoms interacting with the same set of quantum fields. In the following, we shall study in detail the conditions that allow the two otherwise indepedent atoms to become initially entangled through the action of the environment, paying special attention to the external controllable parameters, the bath temperature and the spatial distance ℓ between the two atoms. We shall see that for fixed, finite ℓ, there is always a temperature below which entanglement generation occurs as soon as time starts to become nonzero. Remarkably, it is found that for vanishing ℓ the entanglement thus generated persists even in the long-time asymptotic equilibrium state.

2 Two Atom Master Equation

We shall deal with a system composed by two, identical two-level atoms, that start interacting at time $t = 0$ with a collection of independent, massless, scalar quantum fields at temperature T. We are not interested in the details of the atoms internal dynamics. We shall therefore model them, in a nonrelativistic way, as simple two-level systems, which can be fully described in terms of a two-dimensional Hilbert space.

In absence of any interaction with the external fields, the single atom internal dynamics will be driven by a 2×2 hamiltonian matrix, that in a given basis can be taken to assume

the most general form: $\frac{\omega}{2}\vec{n}\cdot\vec{\sigma}\equiv\frac{\omega}{2}\sum_{i=1}^{3}n_i\sigma_i$, where σ_i, $i=1,2,3$ are the Pauli matrices, n_i, $i=1,2,3$ are the components of a unit vector, while ω represents the gap between the two energy eigenvalues. Then, the atom Hamiltonian H_S is the sum of two such terms:

$$H_S = H_S^{(1)} + H_S^{(2)} , \quad H_S^{(\alpha)} = \frac{\omega}{2}\sum_{i=1}^{3} n_i\,\sigma_i^{(\alpha)} , \quad \alpha = 1,2 , \tag{1}$$

where $\sigma_i^{(1)} = \sigma_i\otimes\mathbf{1}$ and $\sigma_i^{(2)} = \mathbf{1}\otimes\sigma_i$ are the basis operators pertaining to the two different atoms.

As mentioned in the Introduction, the interaction of the atoms with the external fields is assumed to be weak; it can then be described by an hamiltonian H' that is linear in both atom and field variables:

$$H' = \sum_{i=1}^{3}\left(\sigma_i^{(1)}\otimes\Phi_i[f^{(1)}] + \sigma_i^{(2)}\otimes\Phi_i[f^{(2)}]\right) . \tag{2}$$

The operators $\Phi_i(t,\vec{x})$ represent the external quantum fields, taken to be spinless and massless for simplicity. They evolve in time as free relativistic fields with a standard Hamiltonian H_Φ. The atoms are assumed to have a spatial extension described by the two functions $f^{(\alpha)}(\vec{x})$, $\alpha = 1,2$, taken to have a common profile $f(\vec{x})$. To be more specific, we shall choose for the atoms a spherically symmetric shape of infinitesimal size ε:

$$f(\vec{x}) = \frac{1}{\pi^2}\frac{(\varepsilon/2)}{\left[|\vec{x}|^2 + (\varepsilon/2)^2\right]^2} . \tag{3}$$

Further, without loss of generality, the first atom can be positioned at the origin of the reference frame, so that one can assume $f^{(1)}(\vec{x}) \equiv f(\vec{x})$, while the second is displaced by an amount $\vec{\ell}$ with respect to it, and therefore: $f^{(2)}(\vec{x}) = f(\vec{x}+\vec{\ell})$. Since the atom-field interaction takes place on the whole region occupied by the atoms, the field operators entering the interaction Hamiltonian above are smeared over the atom size:

$$\Phi_i[f^{(\alpha)}] = \int d^3x\, f^{(\alpha)}(\vec{x})\,\Phi_i(0,\vec{x}) , \quad \alpha = 1,2 . \tag{4}$$

The total Hamiltonian H describing the complete system, the two atoms together with the external fields Φ_i, can thus be written as

$$H = H_S + H_\Phi + \lambda H' \equiv H_0 + \lambda H' , \tag{5}$$

with λ a small coupling constant. It generates the evolution in t of the corresponding total density matrix ρ_{tot}, $\partial_t\rho_{\text{tot}}(t) = -i[H,\,\rho_{\text{tot}}(t)]$, starting at $t = 0$ from the initial configuration: $\rho_{\text{tot}}(0)$; we shall assume the atom and the fields to be initially prepared in an uncorrelated state, with the fields in the temperature state ρ_β and the atoms in a generic initial state $\rho(0)$, so that $\rho_{\text{tot}}(0) = \rho(0)\otimes\rho_\beta$.

At this point, being interested in studying the dynamics of the two atoms, one conveniently integrates over the unobserved field degrees of freedom and concentrate on the analysis of the reduced time evolution, formally given by the transformation map: $\rho(0) \mapsto \rho(t) \equiv \text{Tr}_\Phi[\rho_{\text{tot}}(t)]$.

The derivation of a physically consistent master equation for the reduced density matrix $\rho(t) \equiv \text{Tr}_\Phi[\rho_{\text{tot}}(t)]$ is notoriously tricky, requiring an *a priori* unambigous separation between subsytem and environment, besides a sufficiently weak interaction between the two [1]-[3]. Generally speaking, this distinction can be achieved when the correlations in the environment decay much faster than the characteristic evolution time of the subsystem alone, given by the inverse of its typical energy scale. In such a case, in the limit of weak couplings, the changes in the evolution of the subsystem occur on time scales that are very long, so large that the details of the internal environment dynamics result irrelevant. This is precisely the situation that occurs for the system under study: indeed, in typical instances, the differences between the atomic internal energy levels result much smaller than the field correlation decay constants so that a clear distinction between subsystem and environment is authomatically achieved.

In practice, the dynamics of the reduced system is obtained by suitably rescaling the time variable, $t \to t/\lambda^2$ and then taking the limit $\lambda \to 0$, following the mathematically precise procedure of the *weak coupling limit* [8],[1]-[3]. The reduced density matrix $\rho(t)$ is then found to obey the following evolution equation:

$$\frac{\partial \rho(t)}{\partial t} = L_{H_S}[\rho(t)] + \mathcal{D}^\sharp[\rho(t)] \,, \qquad L_{H_S}[\rho] \equiv -i[H_S, \rho] \,, \tag{6}$$

where

$$\mathcal{D}^\sharp[\,\cdot\,] = -\lim_{T\to\infty} \frac{1}{T} \int_0^T ds\, \mathcal{U}(-s)\, \mathcal{D}\, \mathcal{U}(s)\,[\,\cdot\,] \,, \qquad \mathcal{U}(s) = e^{s L_{H_S}} \,, \tag{7}$$

and

$$\mathcal{D}[\rho] = \int_0^\infty dt\, \text{Tr}\Big([e^{iH_0 t}\, H'\, e^{-iH_0 t}, [H', \rho \otimes \rho_\beta]]\Big) \,. \tag{8}$$

For the case at hand, the integrals in (7) and (8) can be explicitly computed and the master equation for $\rho(t)$ written down without any ambiguity. It takes a Kossakowski-Lindblad form [17, 18]

$$\frac{\partial \rho(t)}{\partial t} = -i[H_{\text{eff}}, \rho(t)] + \mathcal{L}[\rho(t)] \,, \tag{9}$$

with

$$H_{\text{eff}} = H_S - \frac{i}{2} \sum_{\alpha,\beta=1}^{2} \sum_{i,j=1}^{3} H_{ij}^{(\alpha\beta)}\, \sigma_i^{(\alpha)}\, \sigma_j^{(\beta)} \,, \tag{10}$$

and

$$\mathcal{L}[\rho] = \frac{1}{2} \sum_{\alpha,\beta=1}^{2} \sum_{i,j=1}^{3} C_{ij}^{(\alpha\beta)}[2\, \sigma_j^{(\beta)} \rho\, \sigma_i^{(\alpha)} - \sigma_i^{(\alpha)} \sigma_j^{(\beta)}\, \rho - \rho\, \sigma_i^{(\alpha)} \sigma_j^{(\beta)}] \,. \tag{11}$$

4

The coefficients of the Kossakowski matrix $C_{ij}^{(\alpha\beta)}$ and of the effective Hamiltonian H_{eff} are determined by the field correlation functions in the thermal state ρ_β:

$$G_{ij}^{(\alpha\beta)}(t - t') = \int d^3x \, d^3y \, f^{(\alpha)}(\vec{x}) \, f^{(\beta)}(\vec{y}) \, \langle \Phi_i(t, \vec{x}) \Phi_j(t', \vec{y}) \rangle \ , \tag{12}$$

through their Fourier,

$$\mathcal{G}_{ij}^{(\alpha\beta)}(z) = \int_{-\infty}^{\infty} dt \, e^{izt} \, G_{ij}^{(\alpha\beta)}(t) \ , \tag{13}$$

and Hilbert transform,

$$\mathcal{K}_{ij}^{(\alpha\beta)}(z) = \int_{-\infty}^{\infty} dt \, \text{sign}(t) \, e^{izt} \, G_{ij}^{(\alpha\beta)}(t) = \frac{P}{\pi i} \int_{-\infty}^{\infty} dw \, \frac{\mathcal{G}_{ij}^{(\alpha\beta)}(w)}{w - z} \ , \tag{14}$$

respectively (P indicates principle value). More specifically, one finds:

$$C_{ij}^{(\alpha\beta)} = \sum_{\xi=+,-,0} \sum_{k,l=1}^{3} \mathcal{G}_{kl}^{(\alpha\beta)}(\xi\omega) \, \psi_{ki}^{(\xi)} \, \psi_{lj}^{(-\xi)} \ , \tag{15}$$

and similarly for $H_{ij}^{(\alpha\beta)}$, with $\mathcal{G}_{kl}^{(\alpha\beta)}(\xi\omega)$ replaced by $\mathcal{K}_{kl}^{(\alpha\beta)}(\xi\omega)$, where

$$\psi_{ij}^{(0)} = n_i \, n_j \ , \qquad \psi_{ij}^{(\pm)} = \frac{1}{2}(\delta_{ij} - n_i \, n_j \pm i\epsilon_{ijk} n_k) \ . \tag{16}$$

are the components of auxiliary three-dimensional tensors.[1] Being the sum of three positive terms, the matrix $C_{ij}^{(\alpha\beta)}$ turns out to be positive, so that the dynamical semigroup generated by (9) is composed by completely positive maps: this is the result of adopting a mathematically well-defined formalism [8]. On the other hand, let us remark that direct use of the standard second order perturbative approximation (*e.g.* see [6, 7]) often leads to physically inconsistent results, giving a finite time evolution for $\rho(t)$ that in general does not preserve the positivity of probabilities.

The expressions in (13) and (14) can be explicitly computed by noting that the fields are taken to be independent and assumed to obey a free evolution, so that:

$$\langle \Phi_i(x) \Phi_j(y) \rangle \equiv \text{Tr}[\Phi_i(x) \Phi_j(y) \rho_\beta] = \delta_{ij} \, G(x - y) \ , \tag{17}$$

where $G(x - y)$ is the standard four-dimensional Wightmann function for a single relativistic scalar field in a state at inverse temperature β, formally given by:

$$G(x) = \int \frac{d^4k}{(2\pi)^3} \theta(k^0) \, \delta(k^2) \Big[(1 + \mathcal{N}(k^0)) \, e^{-ik\cdot x} + \mathcal{N}(k^0) \, e^{ik\cdot x} \Big] \ , \tag{18}$$

where

$$\mathcal{N}(k^0) = \frac{1}{e^{\beta k^0} - 1} \ . \tag{19}$$

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[1] We omit the details of the derivation and refer to the Appendix of [16] for an outline of the needed techniques.

Note that, as a result, the correlations in (12) involve $\hat{f}(\vec{k}) = \int d^3x \, e^{i\vec{k}\cdot\vec{x}} \, f(\vec{x})$, namely the Fourier transform of the shape function $f(\vec{x})$ in (3); it can be easily computed to be $\hat{f}(\vec{k}) = e^{-|\vec{k}|\varepsilon/2}$. Being a function of the modulus $|\vec{k}|$ only, this contribution can be conveniently attached to the definition of the Wightmann function $G(x)$, so that the integrand in (18) gets an extra $e^{-\varepsilon k^0}$ overall factor. This damping term assures now the convergence of the integral (18) and corresponds to the usual $i\varepsilon$ prescription for the Wightmann function; in the present setting, it arise as a remnant of the (infinitesimal) size of the atoms.

The behaviour of the field correlations in (12) is also crucial for assuring the convergence of the evolution equation of the reduced density matrix $\rho(t)$ to the limit (6); indeed, one shows [8] that such limit exists only when the combination $|G_{ij}^{(\alpha\beta)}(t)|(1+t)^\eta$ is integrable on the positive half real line, for some $\eta > 0$. In the case of massless fields considered here, this condition is assured by the $1/t^2$ fall off at infinity of the Wightmann function.

Using (17) and (18), the Fourier transform in (13) can now be explicitly evaluated (in the limit of vanishing ε):

$$\mathcal{G}_{ij}^{(\alpha\beta)}(z) = \delta_{ij} \, \mathcal{G}^{(\alpha\beta)}(z) \, , \tag{20}$$

with:

$$\mathcal{G}^{(11)}(z) = \mathcal{G}^{(22)}(z) = \frac{1}{2\pi} \frac{z}{1 - e^{-\beta z}} \, ,$$

$$\mathcal{G}^{(12)}(z) = \mathcal{G}^{(21)}(z) = \frac{1}{2\pi} \frac{z}{1 - e^{-\beta z}} \frac{\sin(\ell z)}{\ell z} \, , \tag{21}$$

where ℓ denotes the modulus of the displacement vector $\vec{\ell}$; then, recalling (14), for the Hilbert transform one similarly finds:

$$\mathcal{K}_{ij}^{(\alpha\beta)}(z) = \delta_{ij} \, \mathcal{K}^{(\alpha\beta)}(z) \, , \qquad \mathcal{K}^{(\alpha\beta)}(z) = \frac{P}{\pi i} \int_{-\infty}^{\infty} dw \, \frac{\mathcal{G}^{(\alpha\beta)}(w)}{w - z} \, . \tag{22}$$

With these results, The Kossakowski matrix can finally be written as:

$$C_{ij}^{(11)} = C_{ij}^{(22)} = A \, \delta_{ij} - iB \, \epsilon_{ijk} \, n_k + C \, n_i \, n_j \, ,$$

$$C_{ij}^{(12)} = C_{ij}^{(21)} = A' \, \delta_{ij} - iB' \, \epsilon_{ijk} \, n_k + C' \, n_i \, n_j \, , \tag{23}$$

where the quantities A, B, C, A', B' and C' depend on the system frequency ω, the inverse temperature β and the separation ℓ between the two atoms:

$$A = \frac{\omega}{4\pi} \left[\frac{1 + e^{-\beta\omega}}{1 - e^{-\beta\omega}} \right] \, , \qquad A' = \frac{\omega}{4\pi} \left[\frac{1 + e^{-\beta\omega}}{1 - e^{-\beta\omega}} \right] \frac{\sin(\omega\ell)}{\omega\ell} \, , \tag{24}$$

$$B = \frac{\omega}{4\pi} \, , \qquad B' = \frac{\omega}{4\pi} \frac{\sin(\omega\ell)}{\omega\ell} \, , \tag{25}$$

$$C = \frac{\omega}{4\pi} \left[\frac{2}{\beta\omega} - \frac{1 + e^{-\beta\omega}}{1 - e^{-\beta\omega}} \right] \, , \qquad C' = \frac{\omega}{4\pi} \left[\frac{2}{\beta\omega} - \frac{1 + e^{-\beta\omega}}{1 - e^{-\beta\omega}} \frac{\sin(\omega\ell)}{\omega\ell} \right] \, . \tag{26}$$

On the other hand, the effective Hamiltonian H_{eff} naturally splits into two contributions: $H_{\text{eff}} = \tilde{H}_S + H_{\text{eff}}^{(12)}$. The first one has the same form of the starting system Hamiltonian in (1), but with a redefined frequency:

$$\tilde{\omega} = \omega + i[\mathcal{K}^{(11)}(-\omega) - \mathcal{K}^{(11)}(\omega)] , \tag{27}$$

while the second one corresponds to an environment generated direct coupling among the two atoms:

$$H_{\text{eff}}^{(12)} = -\frac{i}{2} \sum_{i,j=1}^{3} \Big([\mathcal{K}^{(12)}(-\omega) + \mathcal{K}^{(12)}(\omega)] \, \delta_{ij}$$
$$+ [2\mathcal{K}^{(12)}(0) - \mathcal{K}^{(12)}(-\omega) - \mathcal{K}^{(12)}(\omega)] \, n_i n_j \Big) \, \sigma_i \otimes \sigma_j . \tag{28}$$

These results for the Hamiltonian contributions require some further comments. Recalling (21), the definition of $\mathcal{K}^{(11)}(z)$ in (22) can be split as (similar results hold also for $\mathcal{K}^{(12)}(z)$):

$$\mathcal{K}^{(11)}(z) = \frac{1}{2\pi^2 i} \Bigg[P \int_0^\infty dw \, \frac{w}{w - z} \tag{29}$$

$$+ P \int_0^\infty dw \, \frac{w}{1 - e^{\beta w}} \left(\frac{1}{w + z} - \frac{1}{w - z} \right) \Bigg] , \tag{30}$$

into a vacuum and a temperature-dependent piece. Although not expressible in terms of elementary functions, the temperature dependent second term is a finite, odd function of z, vanishing as β becomes large, *i.e.* in the limit of zero temperature (and as such, it does not contribute to $H_{\text{eff}}^{(12)}$ in (28)). The first contribution in (29) is however divergent. As a consequence, despite some cancellations that occur in (27) and (28), the effective Hamiltonian H_{eff} turns out in general to be infinite, and its definition requires the introduction of a suitable cutoff and a renormalization procedure.

This is a well known fact, and has nothing to do with the weak-coupling assumptions used in deriving the master equation. Rather, the appearance of the divergences is due to the non-relativistic treatment of the two-level atoms, while any sensible calculation of energy shifts would have required the use of quantum field theory techniques [5].

In our quantum mechanical setting, the procedure needed to make H_{eff} well defined is therefore clear: perform a suitable temperature independent subtraction, so that the expressions in (27) and (28) reproduce the correct quantum field theory result, obtained by considering the fields in the vacuum state.

In the following, we shall be interested in analyzing the temperature dependent effects described by the master equation (9); all standard, vacuum generated Hamiltonian contributions will be therefore ignored.

3   Environment Induced Entanglement Generation

Using the explicit form for the master equation derived in the previous Section, one can now investigate whether the thermal bath made of free fields can actually entangle the two

indepedent atoms. Since we are dealing with a couple of two-level systems, this can be achieved with the help of the partial transposition criterion [19, 20]: a two-atom state $\rho(t)$ results entangled at time t if and only if the operation of partial transposition does not preserve its positivity.

We shall first consider the possibility of entanglement creation at the beginning of the evolution. Without loss of generality, one can limit the considerations to pure, separable initial states, and therefore take:

$$\rho(0) = |\varphi\rangle\langle\varphi| \otimes |\psi\rangle\langle\psi| \; ; \tag{31}$$

indeed, if the environment is unable to create entanglement out of pure states, it will certainly not entangle their mixtures. Then, let us examine the behavior in a neighborhood of $t = 0$ of the quantity

$$\mathcal{Q}(t) = \langle\chi|\, \tilde{\rho}(t)\, |\chi\rangle \; , \tag{32}$$

where the tilde signifies partial transposition, *e.g.* with respect to the second factor, and $|\chi\rangle$ is any 4-dimensional vector. The two atoms, initially prepared in a state $\rho(0) \equiv \tilde{\rho}(0)$ as in (31), will surely become entangled if there exists a suitable vector $|\chi\rangle$, such that: *i)* $\mathcal{Q}(0) = 0$ and *ii)* $\partial_t \mathcal{Q}(0) < 0$. In fact, when $\partial_t \mathcal{Q}(0) > 0$ for all choices of the initial state $\rho(0)$ and probe vector $|\chi\rangle$, entanglement can not be generated by the environment, since $\tilde{\rho}$ remains positive. Clearly, the vector $|\chi\rangle$ need be chosen entangled, since otherwise $\mathcal{Q}$ is never negative.

Note that study of the behavior of the quantity $\mathcal{Q}(t)$ near $t = 0$ allows a very explicit analysis of entanglement generation. Indeed, $\partial_t \mathcal{Q}(0)$ can be easily computed through the time derivative $\partial_t \tilde{\rho}(0)$, that in turn can be obtained by taking the partial transposition of the r.h.s. of (9) (with H_{eff} set to zero, as explained above). In this way, one can construct a test of entanglement creation, valid for any probe vector $|\chi\rangle$.

In order to show this, consider first the orthonormal basis $\{|\varphi\rangle, |\tilde{\varphi}\rangle\}$, $\{|\psi\rangle, |\tilde{\psi}\rangle\}$, obtained by augmenting with the two states $|\tilde{\varphi}\rangle$ and $|\tilde{\psi}\rangle$ the ones that define $\rho(0)$ in (31). They can be both unitarily rotated to the standard basis $\{|-\rangle, |+\rangle\}$ of σ_3:

$$\begin{aligned}
|\varphi\rangle &= U|-\rangle & |\tilde{\varphi}\rangle &= U|+\rangle \; , \\
|\psi\rangle &= V|-\rangle & |\tilde{\psi}\rangle &= V|+\rangle \; .
\end{aligned} \tag{33}$$

Similarly, the unitary transformations U and V induce orthogonal transformations $\mathcal{U}$ and $\mathcal{V}$, respectively, on the Pauli matrices:

$$U^\dagger \sigma_i U = \sum_{j=1}^{3} \mathcal{U}_{ij}\sigma_j \; , \quad V^\dagger \sigma_i V = \sum_{j=1}^{3} \mathcal{V}_{ij}\sigma_j \; . \tag{34}$$

Direct computation then shows that $\partial_t \mathcal{Q}(0)$ can be written as a quadratic form in the independent components of the probe vector $|\chi\rangle$, with coefficients that involve the four 3×3 matrices $C^{(11)}$, $C^{(22)}$, $C^{(12)}$, $C^{(21)}$ that form the Kossakowski coefficients given in (15). As a consequence, vectors $|\chi\rangle$ exist making this form negative, *i.e.* $\partial_t \mathcal{Q}(0) < 0$, if and only if its corresponding discriminant is negative; explicitly:

$$\langle u|C^{(11)}|u\rangle \, \langle v|(C^{(22)})^T|v\rangle < |\langle u|\mathcal{R}e(C^{(12)})|v\rangle|^2 \; , \tag{35}$$

where T means matrix transposition. The three-dimensional vectors $|u\rangle$ and $|v\rangle$ contain the information about the starting factorized state (31): their components can be in fact expressed as:

$$u_i = \sum_{j=1}^{3} \mathcal{U}_{ij} \langle +|\sigma_j|-\rangle , \quad v_i = \sum_{j=1}^{3} \mathcal{V}_{ij} \langle -|\sigma_j|+\rangle . \tag{36}$$

Therefore, the external quantum fields will be able to entangle the two atoms evolving with the Markovian dynamics generated by (9) and characterized by the Kossakowski matrix (15), if there exists an initial state $|\varphi\rangle\langle\varphi| \otimes |\psi\rangle\langle\psi|$, or equivalently orthogonal transformations $\mathcal{U}$ and $\mathcal{V}$, for which the inequality (35) is satisfied.

That this is indeed the case for the matrices $C^{(\alpha\beta)}$ in (23) can be easily shown. First note that, without loss of generality, the unit vector $\vec{n}$ that defines the internal atom Hamiltonian in (1) can be oriented along the third axis. Consider then the initial state $\rho(0) = |-\rangle\langle-| \otimes |+\rangle\langle+|$, constructed out of the eigenstates of the single atom Hamiltonian. Recalling the definitions (31) and (33), one then finds $U = \mathbf{1}$, so that the three-dimensional vector $|u\rangle$ has components $u_i = \{1, -i, 0\}$, and further $v_i = u_i$. Then, the inequality (35) reduces to:

$$R^2 + S^2 > 1 , \qquad R \equiv \frac{B}{A} = \frac{1 - e^{-\beta\omega}}{1 + e^{-\beta\omega}} , \qquad S = \frac{\sin(\omega\ell)}{\omega\ell} . \tag{37}$$

Although both R and S take values in the interval $[0, 1]$, one can easily make the sum of their squares to exceed unity by adjusting the inverse temperature β and the atom separation ℓ. In particular, for a given, finite separation ℓ, one can always find a temperature below which the inequality in (37) is satisfied and therefore entanglement created. The case of vanishing separation is even more striking, since the inequality (37) reduce to $R > 0$, which is always satisfied, except in the limit of an infinite bath temperature.

Analogous results hold also in the case of a zero temperature bath. Indeed, in this case $R = 1$, and entanglement is generated for any finite separation of the two atoms (similar conclusions have also been reported before in Ref.[21]).

The role played by the two considered control parameters, the bath temperature and the atom separation, in triggering entanglement creation is now apparent. The temperature of the external environment determines the amount of noise that is induced in the dynamics of the two independent atoms. Noise is known to reduce quantum correlations, and indeed, the higher the temperature, the less effective is the entanglement power of the bath. Environment induced entanglement generation is nevertheless a robust phenomenon: it always occurs except in the limit of an infinitely large temperature.

A similar role is played by the second control parameter, the spacial atom separation: entanglement enhancement is more effective the closer the two atoms are, and turns out to be impossible only for an infinitely large separation. The interplay between the effects of these two control parameters is neatly summarized by the inequality (37).

To our knowledge, this is the first instance of open quantum system control through the bath parameters, and not via the subsystem Hamiltonian [22, 23]. This approach, which

might prove very fruitful in the field of quantum information, is still in a very preliminary stage; further developments are presently under study and will be reported elsewhere.

Finally, note that the above choice for $\rho(0)$ is not restrictive: one can always use the transformations in (33) to map it to the generic initial state (31), at the expences of using suitably rotated $|u\rangle$ and $|v\rangle$, as given in (36). As a result, the expression of the entanglement test in (35) becomes more complicated than in (37), but the final conclusions remain unchanged.

4   Asymptotic Entanglement

On the basis of the anlysis presented in the previous Section two atoms, initially prepared in a separate state, will generically become entangled as a result of their independent interaction with a bath made of thermal quantum fields: quantum correlations are generated among the two atoms as soon as $t > 0$. The test in (35), on which this conclusion is based, is however unable to determine the fate of this quantum correlations, as time becomes large.

On general grounds, one expects that the effects of decoherence and dissipation that counteract entanglement production be dominant at large times, so that no entanglement is left in the end. This is precisely the conclusion that one obtains by a careful analysis of the structure of the dynamics generated by the master equation in (9), with Kossakowski coefficients as in (23)-(26): the asymptotic equilibrium state for the two atoms turns out to be always separate for ℓ finite (a detailed account of this result is beyond the scope of the present work, and will be reported elsewhere).

However, the case of a vanishing atom separation is again special and deserves a closer examination. Note that in such a situation, the matrices in (23) become all equal $C^{(11)} = C^{(22)} = C^{(12)} = C^{(21)}$. This particular choice for the Kossakowski matrix is also adopted in the description of the phenomenon of resonance fluorescence [24, 7]. Therefore, the discussion of the vanishing ℓ limit is of relevance also from the phenomenological point of view.

The presence of an equilibrium state ρ^∞ can be in general determined by setting to zero the r.h.s. of the evolution equation (9). As previously explained, we shall ignore the Hamiltonian piece since it can not give rise to temperature dependent entanglement phenomena, and concentrate on the study of the effects induced by the dissipative part; the equilibrium condition reduces then to $\mathcal{L}[\rho^\infty] = 0$. Direct computation leads to the following result:

$$\rho^\infty = \frac{1}{4}\left[\mathbf{1} \otimes \mathbf{1} - a \sum_{i=1}^{3} n_i(\mathbf{1} \otimes \sigma_i + \sigma_i \otimes \mathbf{1}) + \sum_{i,j=1}^{3} (b\,\delta_{ij} + c\,n_i n_j)\,\sigma_i \otimes \sigma_j\right], \qquad (38)$$

with

$$a = \frac{R}{3+R^2}(\tau + 3)\,, \qquad b = \frac{\tau - R^2}{3+R^2}(\tau + 3)\,, \qquad c = R\,a\,. \qquad (39)$$

Here, $R = B/A$ is the temperature dependent ratio already introduced in (37), while the parameter $\tau = \frac{1}{4}\sum_{i=1}^{3}\mathrm{Tr}[\rho(0)\,(\sigma_i \otimes \sigma_i)]$ contains the dependence on the initial state (positivity of $\rho(0)$ requires $-3 \leq \tau \leq 1$): the equilibrium state is therefore not unique. In spite of this, remarkably, the asymptotic state (38) turns out to be still entangled.

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To explicitly show this, one can as before act with the operation of partial transposition on ρ^∞ to see whether negative eigenvalues are present. Alternatively, one can resort to one of the available entanglement measures and concurrence appears here to be the more appropriate: its value $\mathcal{C}[\rho]$ ranges from zero, for separable states, to one, for fully entangled states, like the Bell states [25]-[27]. In the case of the state ρ^∞ above, one finds

$$\mathcal{C}[\rho^\infty] = \max\left\{ \frac{(3-R^2)}{2(3+R^2)}\left[\frac{5R^2-3}{3-R^2} - \tau\right], \, 0 \right\} . \tag{40}$$

This expression is indeed nonvanishing, provided we start with an initial state $\rho(0)$ for which

$$\tau < \frac{5R^2-3}{3-R^2} . \tag{41}$$

The concurrence is therefore a linearly decreasing function of τ, starting from its maximum $\mathcal{C}[\rho^\infty] = 1$ for $\tau = -3$ and reaching zero at $\tau = (5R^2 - 3)/(3 - R^2)$.

This result is remarkable, since it implies that the dynamics in (9) not only can initially generate entanglement: it can continue to enhance it even in the asymptotic long time regime.[2] In other terms, put the two atoms in the same place and prepare them at $t = 0$ in a separable state; then, provided the condition (41) is satisfied, their long time equilibrium state will turn out to be entangled.

The simplest example of a separable state is provided by the direct product of pure states as in (31). When $|\varphi\rangle$ and $|\psi\rangle$ are orthogonal, so that $\tau = -1$, one explicitly finds: $\mathcal{C}[\rho^\infty] = \max\{2R^2/(3 + R^2), 0\}$. Notice that $\mathcal{C}[\rho^\infty]$ reaches its maximum value of $1/2$ when $R = 1$, *i.e.* at zero temperature, while it vanishes when the temperature becomes infinitely large, *i.e.* $R = 0$. As already remarked, this has to be expected, since in this case the decoherence effects of the bath become dominant.

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